

# Dynamics of symmetric holomorphic maps on projective spaces

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## Abstract

We consider complex dynamics of a *critically finite* holomorphic map from  $\mathbf{P}^k$  to  $\mathbf{P}^k$ , which has symmetries associated with the symmetric group  $S_{k+2}$  acting on  $\mathbf{P}^k$ , for each  $k \geq 1$ . The Fatou set of each map of this family consists of attractive basins of superattracting points. Each map of this family satisfies Axiom A.

## 1 Introduction

For a finite group  $G$  acting on  $\mathbf{P}^k$  as projective transformations, we say that a rational map  $f$  on  $\mathbf{P}^k$  is  $G$ -equivariant if  $f$  commutes with each element of  $G$ . That is,  $f \circ r = r \circ f$  for any  $r \in G$ , where  $\circ$  denotes the composition of maps. Doyle and McMullen [4] introduced the notion of *equivariant* functions on  $\mathbf{P}^1$  to solve quintic equations. See also [11] for *equivariant* functions on  $\mathbf{P}^1$ . Crass [2] extended Doyle and McMullen's algorithm to higher dimensions to solve sextic equations. Crass [3] found a good family of finite groups and *equivariant* maps for which one may say something about global dynamics. Crass [3] conjectured that the Fatou set of each map of this family consists of attractive basins of superattracting points. Although I do not know whether this family has relation to solving equations or not, our results will give affirmative answers for the conjectures in [3].

In section 2 we shall explain an action of the symmetric group  $S_{k+2}$  on  $\mathbf{P}^k$  and properties of our  $S_{k+2}$ -equivariant map. In section 3 and 4 we shall show our results about the Fatou sets and hyperbolicity of our maps by using properties of our maps and Kobayashi metrics.

## 2 $S_{k+2}$ -equivariant maps

Crass [3] selected the symmetric group  $S_{k+2}$  as a finite group acting on  $\mathbf{P}^k$  and found an  $S_{k+2}$ -equivariant map which is holomorphic and *critically finite* for each  $k \geq 1$ . We denote by  $C = C(f)$  the critical set of  $f$  and say that  $f$  is *critically finite* if each irreducible component of  $C(f)$  is periodic or preperiodic. More precisely,  $S_{k+2}$ -equivariant map  $g_{k+3}$  defined in section 2.2 preserves each irreducible component of  $C(g_{k+3})$ , which is a projective hyperplane. The complement of  $C(g_{k+3})$  is Kobayashi hyperbolic. Furthermore restrictions of  $g_{k+3}$  to invariant projective subspaces have the same properties as above. See section 2.3 for details.

### 2.1 $S_{k+2}$ acts on $\mathbf{P}^k$

An action of the  $(k+2)$ -th symmetric group  $S_{k+2}$  on  $\mathbf{P}^k$  is induced by the permutation action of  $S_{k+2}$  on  $\mathbf{C}^{k+2}$  for each  $k \geq 1$ . The transposition  $(i, j)$  in  $S_{k+2}$  corresponds with the transposition " $u_i \leftrightarrow u_j$ " on  $\mathbf{C}_u^{k+2}$ , which pointwise fixes the hyperplane  $\{u_i = u_j\} = \{u \in \mathbf{C}_u^{k+2} \mid u_i = u_j\}$ . Here  $\mathbf{C}^{k+2} = \mathbf{C}_u^{k+2} = \{u = (u_1, u_2, \dots, u_{k+2}) \mid u_i \in \mathbf{C} \text{ for } i = 1, \dots, k+2\}$ .

The action of  $S_{k+2}$  preserves a hyperplane  $H$  in  $\mathbf{C}_u^{k+2}$ , which is identified with  $\mathbf{C}_x^{k+1}$  by projection  $A : \mathbf{C}_u^{k+2} \rightarrow \mathbf{C}_x^{k+1}$ ,

$$H = \left\{ \sum_{i=1}^{k+2} u_i = 0 \right\} \stackrel{A}{\cong} \mathbf{C}_x^{k+1} \text{ and } A = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}.$$

Here  $\mathbf{C}^{k+1} = \mathbf{C}_x^{k+1} = \{x = (x_1, x_2, \dots, x_{k+1}) \mid x_i \in \mathbf{C} \text{ for } i = 1, \dots, k+1\}$ .

Thus the permutation action of  $S_{k+2}$  on  $\mathbf{C}_u^{k+2}$  induces an action of " $S_{k+2}$ " on  $\mathbf{C}_x^{k+1}$ . Here " $S_{k+2}$ " is generated by the permutation action  $S_{k+1}$  on  $\mathbf{C}_x^{k+1}$  and a  $(k+1, k+1)$ -matrix  $T$  which corresponds to the transposition  $(1, k+2)$  in  $S_{k+2}$ ,

$$T = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \dots & 1 \end{pmatrix}.$$

Hence the hyperplane corresponding to  $\{u_i = u_j\}$  is  $\{x_i = x_j\}$  for  $1 \leq i < j \leq k+1$ . The hyperplane corresponding to  $\{u_i = u_{k+2}\}$  is  $\{x_i = 0\}$  for  $1 \leq i \leq k+1$ . Each element in " $S_{k+2}$ " which corresponds to some transposition in  $S_{k+2}$  pointwise fixes one of these hyperplanes in  $\mathbf{C}_x^{k+1}$ .

The action of " $S_{k+2}$ " on  $\mathbf{C}^{k+1}$  projects naturally to the action of " $S_{k+2}$ " on  $\mathbf{P}^k$ . These hyperplanes on  $\mathbf{C}^{k+1}$  projects naturally to projective hyperplanes on  $\mathbf{P}^k$ . Here  $\mathbf{P}^k = \{x = [x_1 : x_2 : \cdots : x_{k+1}] \mid (x_1, x_2, \cdots, x_{k+1}) \in \mathbf{C}^{k+1} \setminus \{0\}\}$ . Each element in the action of " $S_{k+2}$ " on  $\mathbf{P}^k$  which corresponds to some transposition in  $S_{k+2}$  pointwise fixes one of these projective hyperplanes. We denote " $S_{k+2}$ " also by  $S_{k+2}$  and call these projective hyperplanes transposition hyperplanes.

## 2.2 Existence of our maps

One way to get  $S_{k+2}$ -equivariant maps on  $\mathbf{P}^k$  which are *critically finite* is to make  $S_{k+2}$ -equivariant maps whose critical sets coincide with the union of the transposition hyperplanes.

**Theorem 1** ([3]). *For each  $k \geq 1$ ,  $g_{k+3}$  defined below is the unique  $S_{k+2}$ -equivariant holomorphic map of degree  $k+3$  which is doubly critical on each transposition hyperplane.*

$$g = g_{k+3} = [g_{k+3,1} : g_{k+3,2} : \cdots : g_{k+3,k+1}] : \mathbf{P}^k \rightarrow \mathbf{P}^k,$$

$$\text{where } g_{k+3,l}(x) = x_l^3 \sum_{s=0}^k (-1)^s \frac{s+1}{s+3} x_l^s A_{k-s}, \quad A_0 = 1,$$

and  $A_{k-s}$  is the elementary symmetric function of degree  $k-s$  in  $\mathbf{C}^{k+1}$ .

Then the critical set of  $g$  coincides with the union of the transposition hyperplanes. Since  $g$  is  $S_{k+2}$ -equivariant and each transposition hyperplane is pointwise fixed by some element in  $S_{k+2}$ ,  $g$  preserves each transposition hyperplane. In particular  $g$  is *critically finite*. Although Crass [3] used this explicit formula to prove Theorem 1, we shall only use properties of the  $S_{k+2}$ -equivariant maps described below.

## 2.3 Properties of our maps

Let us look at properties of the  $S_{k+2}$ -equivariant map  $g$  on  $\mathbf{P}^k$  for a fixed  $k$ , which is proved in [3] and shall be used to prove our results. Let  $L^{k-1}$  denote one of the transposition hyperplanes, which is isomorphic to  $\mathbf{P}^{k-1}$ . Let  $L^m$  denote one of the intersections of  $(k-m)$  or more distinct transposition hyperplanes which is isomorphic to  $\mathbf{P}^m$  for  $m = 0, 1, \cdots, k-1$ .

First, let us look at properties of  $g$  itself. The critical set of  $g$  consists of the union of the transposition hyperplanes. By  $S_{k+2}$ -equivariance,  $g$  preserves each transposition hyperplane. Furthermore the complement of the critical set of  $g$  is Kobayashi hyperbolic.

Next, let us look at properties of  $g$  restricted to  $L^m$  for  $m = 1, 2, \dots, k-1$ . Let us fix any  $m$ . Since  $g$  preserves each  $L^m$ , we can also consider the dynamics of  $g$  restricted to any  $L^m$ . Each restricted map has the same properties as above. Let us fix any  $L^m$  and denote by  $g|_{L^m}$  the restricted map of  $g$  to the  $L^m$ . The critical set of  $g|_{L^m}$  consists of the union of intersections of the  $L^m$  and another  $L^{k-1}$  which does not include the  $L^m$ . We denote it by  $L^{m-1}$ , which is an irreducible component of the critical set of  $g|_{L^m}$ . By  $S_{k+2}$ -equivariance,  $g|_{L^m}$  preserves each irreducible component of the critical set of  $g|_{L^m}$ . Furthermore the complement of the critical set of  $g|_{L^m}$  in  $L^m$  is Kobayashi hyperbolic.

Finally, let us look at a property of superattracting fixed points of  $g$ . The set of superattracting points, where the derivative of  $g$  vanishes for all directions, coincides with the set of  $L^0$ 's.

**Remark 1.** For every  $k \geq 1$  and every  $m$ ,  $1 \leq m \leq k$ , a restricted map of  $g_{k+3}$  to any  $L^m$  is not conjugate to  $g_{m+3}$ .

## 2.4 Examples for $k = 1$ and 2

Let us see transposition hyperplanes of the  $S_3$ -equivariant function  $g_4$  and the  $S_4$ -equivariant map  $g_5$  to make clear what  $L^m$  is. In [3] one can find explicit formulas and figures of dynamics of  $S_{k+2}$ -equivariant maps in low-dimensions.

### 2.4.1 $S_3$ -equivariant function $g_4$ in $\mathbf{P}^1$

$$g_3([x_1 : x_2]) = [x_1^3(-x_1 + 2x_2) : x_2^3(2x_1 - x_2)] : \mathbf{P}^1 \rightarrow \mathbf{P}^1,$$

$$C(g_3) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_2\} = \{0, 1, \infty\} \text{ in } \mathbf{P}^1.$$

In this case "transposition hyperplanes" are points in  $\mathbf{P}^1$  and  $L^0$  denotes one of three superattracting fixed points of  $g_3$ .

### 2.4.2 $S_4$ -equivariant map $g_5$ in $\mathbf{P}^2$

$$C(g_5) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \\ \{x_1 = x_2\} \cup \{x_2 = x_3\} \cup \{x_3 = x_1\} \text{ in } \mathbf{P}^2.$$

In this case  $L^1$  denotes one of six transposition hyperplanes in  $\mathbf{P}^2$ , which is an irreducible component of  $C(g_5)$ . For example, let us fix a transposition hyperplane  $\{x_1 = 0\}$ . Since  $g_5$  preserves each transposition hyperplane,

we can also consider the dynamics of  $g_5$  restricted to  $\{x_1 = 0\}$ . We denote by  $g_5|_{\{x_1=0\}}$  the restricted map of  $g_5$  to  $\{x_1 = 0\}$ . The critical set of  $g_5|_{\{x_1=0\}}$  in  $\{x_1 = 0\} \simeq \mathbf{P}^1$  is

$$C(g_5|_{\{x_1=0\}}) = \{[0 : 1 : 0], [0 : 0 : 1], [0 : 1 : 1]\}.$$

When we use  $L^0$  after we fix  $\{x_1 = 0\}$ ,  $L^0$  denotes one of intersections of  $\{x_1 = 0\}$  and another transposition hyperplane, which is a superattracting fixed point of  $g_5|_{\{x_1=0\}}$  in  $\mathbf{P}^1$ . The set of superattracting fixed points of  $g_5$  in  $\mathbf{P}^2$  is

$$\{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1], [1 : 1 : 0], [1 : 0 : 1], [0 : 1 : 1]\}.$$

In general  $L^0$  denotes one of intersections of two or more transposition hyperplanes, which is a superattracting fixed point of  $g_5$  in  $\mathbf{P}^2$ .

### 3 The Fatou sets of the $S_{k+2}$ -equivariant maps

#### 3.1 Definitions and preliminaries

Let us recall theorems about *critically finite* holomorphic maps. Let  $f$  be a holomorphic map from  $\mathbf{P}^k$  to  $\mathbf{P}^k$ . The Fatou set of  $f$  is defined to be the maximal open subset where the iterates  $\{f^n\}_{n \geq 0}$  is a normal family. The Julia set of  $f$  is defined to be the complement of the Fatou set of  $f$ . Each connected component of the Fatou set is called a Fatou component. Let  $U$  be a Fatou component of  $f$ . A holomorphic map  $h$  is said to be a limit map on  $U$  if there is a subsequence  $\{f^{n_s}|_U\}_{s \geq 0}$  which locally converges to  $h$  on  $U$ . We say that a point  $q$  is a Fatou limit point if there is a limit map  $h$  on a Fatou component  $U$  such that  $q \in h(U)$ . The set of all Fatou limit points is called the Fatou limit set. We define the  $\omega$ -limit set  $E(f)$  of the critical points by

$$E(f) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{n=j}^{\infty} f^n(C)}.$$

**Theorem 2.** ([10, Proposition 5.1]) *If  $f$  is a critically finite holomorphic map from  $\mathbf{P}^k$  to  $\mathbf{P}^k$ , then the Fatou limit set is contained in the  $\omega$ -limit set  $E(f)$ .*

Let us recall the notion of Kobayashi metrics. Let  $M$  be a complex manifold and  $K_M(x, v)$  the Kobayashi quasimetric on  $M$ ,

$$\inf \left\{ |a| \left| \varphi : \mathbf{D} \rightarrow M : \text{holomorphic}, \varphi(0) = x, D\varphi \left( a \left( \frac{\partial}{\partial z} \right)_0 \right) = v, a \in \mathbf{C} \right\}$$

for  $x \in M$ ,  $v \in T_x M$ ,  $z \in \mathbf{D}$ , where  $\mathbf{D}$  is the unit disk in  $\mathbf{C}$ . We say that  $M$  is Kobayashi hyperbolic if  $K_M$  becomes a metric. Theorem 5 is a corollary of Theorem 3 and Theorem 4 for  $k = 1$  and 2.

**Theorem 3.** (a basic result whose former statement can be found in [8, Corollary 14.5]) *If  $f$  is a critically finite holomorphic function from  $\mathbf{P}^1$  to  $\mathbf{P}^1$ , then the only Fatou components of  $f$  are attractive components of superattracting points. Moreover if the Fatou set is not empty, then the Fatou set has full measure in  $\mathbf{P}^1$ .*

**Theorem 4.** ([5, theorem 7.7]) *If  $f$  is a critically finite holomorphic map from  $\mathbf{P}^2$  to  $\mathbf{P}^2$  and the complement of  $C(f)$  is Kobayashi hyperbolic, then the only Fatou components of  $f$  are attractive components of superattracting points.*

### 3.2 Our first result

Let us fix any  $k$  and  $g = g_{k+3}$ . For every  $m$ ,  $2 \leq m \leq k$ , we can apply an argument in [5] to a restricted map of  $g$  to any  $L^m$  because every  $L^{m-1}$  is smooth and because every  $L^m \setminus C(g|_{L^m})$  is Kobayashi hyperbolic. We shall use this argument in Lemma 1, which is used to prove Proposition 1.

**Proposition 1.** *For any Fatou component  $U$  which is disjoint from  $C(g)$ , there exists an integer  $n$  such that  $g^n(U)$  intersects with  $C(g)$ .*

*Proof:* We suppose that  $g^n(U)$  is disjoint from  $C(g)$  for any  $n$  and derive a contradiction by using Lemma 1 and Remark 3 below. Take any point  $x_0 \in U$ . Since  $E(g)$  coincides with  $C(g)$ ,  $g^n(x_0)$  accumulates to  $C(g)$  as  $n$  tends to  $\infty$  from Theorem 2. Since  $C(g)$  is the union of the transposition hyperplanes, there exists a smallest integer  $m_1$  such that  $g^n(x_0)$  accumulates to some  $L^{m_1}$ . Let  $h_1$  be a limit map on  $U$  such that  $h_1(x_0)$  belongs to the  $L^{m_1}$ . From Lemma 1 below, the intersection of  $h_1(U)$  and the  $L^{m_1}$  is an open set in the  $L^{m_1}$  and is contained in the Fatou set of  $g|_{L^{m_1}}$ .

We next consider the dynamics of  $g|_{L^{m_1}}$ . If there exists an integer  $n_2$  such that  $g^{n_2}(h_1(U) \cap L^{m_1})$  intersects with  $C(g|_{L^{m_1}})$ , then  $g^{n_2}(h_1(U) \cap L^{m_1})$  intersects with some  $L^{m_1-1}$ . In this case we can consider the dynamics of  $g|_{L^{m_1-1}}$ . On the other hand, if there does not exist such  $n_2$ , then there exists an integer  $m_2$  and a limit map  $h_2$  on  $h_1(U) \cap L^{m_1}$  such that the intersection of  $h_2(h_1(U) \cap L^{m_1})$  and some  $L^{m_2}$  is an open set in the  $L^{m_2}$  from Remark 3 below. Thus it is contained in the Fatou set of  $g|_{L^{m_2}}$ . Here  $m_2$  is smaller than  $m_1$ . In this case we can consider the dynamics of  $g|_{L^{m_2}}$ .

We continue the same argument above. These reductions finally come to some  $L^1$  and we use Theorem 3. One can find a similar reduction argument in the proof of Theorem 5. Consequently  $g^n(x_0)$  accumulates to some

superattracting point  $L^0$ . So there exists an integer  $s$  such that  $g^s$  sends  $U$  to the attractive Fatou component which contains the superattracting point  $L^0$ . Thus  $g^s(U)$  intersects with  $C(g)$ , which is a contradiction.  $\square$

**Remark 2.** *Even if a Fatou component  $U$  intersects with some  $L^m$  and is disjoint from any  $L^{m-1}$ , then the similar thing as above holds for the dynamics in the  $L^m$ . In this case  $U \cap L^m$  is contained in the Fatou set of  $g|_{L^m}$  and there exists an integer  $n$  such that  $g^n(U \cap L^m)$  intersects with  $C(g|_{L^m})$ .*

**Lemma 1.** *For any Fatou component  $U$  which is disjoint from  $C(g)$  and any point  $x_0 \in U$ , let  $h$  be a limit map on  $U$  such that  $h(x_0)$  belongs to some  $L^m$  and does not belong to any  $L^{m-1}$ . If  $g^n(U)$  is disjoint from  $C(g)$  for every  $n \geq 1$ , then the intersection of  $h(U)$  and the  $L^m$  is an open set in the  $L^m$ .*

*Proof:* Let  $B$  be the complement of  $C(g)$ . Since  $B$  is Kobayashi hyperbolic and  $B$  includes  $g^{-1}(B)$ ,  $g^{-1}(B)$  is Kobayashi hyperbolic, too. So we can use Kobayashi metrics  $K_B$  and  $K_{g^{-1}(B)}$ . Since  $B$  includes  $g^{-1}(B)$ ,

$$K_B(x, v) \leq K_{g^{-1}(B)}(x, v) \text{ for all } x \in g^{-1}(B), v \in T_x \mathbf{P}^k.$$

In addition, since  $g$  is an unbranched covering from  $g^{-1}(B)$  to  $B$ ,

$$K_{g^{-1}(B)}(x, v) = K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B), v \in T_x \mathbf{P}^k.$$

From these two inequalities we have the following inequality

$$K_B(x, v) \leq K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B), v \in T_x \mathbf{P}^k.$$

Since the same argument holds for any  $g^n$  from  $g^{-n}(B)$  to  $B$ ,

$$K_B(x, v) \leq K_B(g^n(x), Dg^n(v)) \text{ for all } x \in g^{-n}(B), v \in T_x \mathbf{P}^k.$$

Since  $g^n$  is an unbranched covering from  $U$  to  $g^n(U)$  and  $B$  includes  $g^n(U)$  for every  $n$ , a sequence  $\{K_B(g^n(x), Dg^n(v))\}_{n \geq 0}$  is bounded for all  $x \in U$ ,  $v \in T_x \mathbf{P}^k$ . Hence we have the following inequality for any unit vectors  $v_n$  in  $T_{x_0} U$  with respect to the Fubini-Study metric in  $\mathbf{P}^k$ ,

$$(1) \quad 0 < \inf_{|v|=1} K_B(x_0, v) \leq K_B(x_0, v_n) \leq K_B(g^n(x_0), Dg^n(x_0)v_n) < \infty.$$

That is, the sequence  $\{K_B(g^n(x_0), Dg^n(x_0)v_n)\}_{n \geq 0}$  is bounded away from 0 and  $\infty$  uniformly.

We shall choose  $v_n$  so that  $Dg^n(x_0)v_n$  keeps parallel to the  $L^m$  and claim that  $Dh(x_0)v \neq \mathbf{0}$  for any accumulation vector  $v$  of  $v_n$ . Let  $h = \lim_{n \rightarrow \infty} g^n$

for simplicity. Let  $V$  be a neighborhood of  $h(x_0)$  and  $\psi$  a local coordinate on  $V$  so that  $\psi(h(x_0)) = \mathbf{0}$  and  $\psi(L^m \cap V) \subset \{y = (y_1, y_2, \dots, y_k) \mid y_1 = \dots = y_{k-m} = 0\}$ . In this chart there exists a constant  $r > 0$  such that a polydisk  $P(\mathbf{0}, 2r)$  does not intersect with any images of transposition hyperplanes which do not include the  $L^m$ . Since  $\psi(g^n(x_0))$  converges to  $\mathbf{0}$  as  $n$  tends to  $\infty$ , we may assume that  $\psi(g^n(x_0))$  belongs to  $P(\mathbf{0}, r)$  for large  $n$ . Let  $\{v_n\}_{n \geq 0}$  be unit vectors in  $T_{x_0} \mathbf{P}^k$  and  $\{w_n\}_{n \geq 0}$  vectors in  $T_{\psi(g^n(x_0))} \mathbf{C}^k$  so that  $w_n$  keep parallel to  $\psi(L^m)$  with a same direction and

$$Dg^n(x_0)v_n = |Dg^n(x_0)v_n| D\psi^{-1}(w_n).$$

So we may assume that the length of  $w_n$  is almost unit for large  $n$ . We define holomorphic maps  $\varphi_n$  from  $\mathbf{D}$  to  $P(\mathbf{0}, 2r)$  as

$$\varphi_n(z) = \psi(g^n(x_0)) + rz w_n \text{ for } z \in \mathbf{D}$$

and consider holomorphic maps  $\psi^{-1} \circ \varphi_n$  from  $\mathbf{D}$  to  $B$  for large  $n$ . Then

$$(\psi^{-1} \circ \varphi_n)(0) = g^n(x_0),$$

$$D(\psi^{-1} \circ \varphi_n) \left( \frac{|Dg^n(x_0)v_n|}{r} \left( \frac{\partial}{\partial z} \right)_0 \right) = Dg^n(x_0)v_n.$$

Suppose  $Dh(x_0)v = \mathbf{0}$ , then  $Dg^n(x_0)v$  converges to  $\mathbf{0}$  as  $n$  tends to  $\infty$  and so does  $Dg^n(x_0)v_n$ . By the definition of Kobayashi metric we have that

$$K_B(g^n(x_0), Dg^n(x_0)v_n) \leq \frac{|Dg^n(x_0)v_n|}{r} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since this contradicts (1), we have  $Dh(x_0)v \neq \mathbf{0}$ . This holds for all directions which are parallel to  $\psi(L^m)$ . Consequently the intersection of  $h(U)$  and the  $L^m$  is an open set in  $L^m$ .  $\square$

**Remark 3.** *The similar thing as above holds for the dynamics of any restricted map. Thus even if a Fatou component  $g^n(U)$  intersects with  $C(g)$  for some  $n$ , the same result as above holds. Because one can consider the dynamics in the  $L^m$  when  $g^n(U)$  intersects with some  $L^m$ .*

**Theorem 5.** *For each  $k \geq 1$ , the Fatou set of the  $S_{k+2}$ -equivariant map  $g$  consists of attractive basins of superattracting fixed points which are intersections of  $k$  or more distinct transposition hyperplanes.*



*Proof:* This theorem follows from Proposition 1 and Remark 2 immediately. Let us describe details. Take any Fatou component  $U$ . From Proposition 1 there exists an integer  $n_k$  such that  $g^{n_k}(U)$  intersects with  $C(g)$ . Since  $C(g)$  is the union of the transposition hyperplanes,  $g^{n_k}(U)$  intersects with some  $L^{k-1}$ . By doing the same thing as above for the dynamics of  $g$  restricted to the  $L^{k-1}$ , there exists an integer  $n_{k-1}$  such that  $g^{n_k+n_{k-1}}(U)$  intersects with some  $L^{k-2}$  from Remark 2. We again do the same thing as above for the dynamics of  $g$  restricted to the  $L^{k-2}$ .

These reductions finally come to some  $L^1$ . That is, there exists integers  $n_{k-2}, \dots, n_2$  such that  $g^{n_k+n_{k-1}+\dots+n_2}(U)$  intersects with some  $L^1$ . From Theorem 3 there exists an integer  $n_1$  such that  $g^{n_1}(g^{n_k+n_{k-1}+\dots+n_2}(U))$  contains some  $L^0$ . Hence  $g^{n_k+n_{k-1}+\dots+n_1}$  sends  $U$  to the attractive Fatou component which contains the superattracting fixed point  $L^0$  in  $\mathbf{P}^k$ .  $\square$

## 4 Axiom A and the $S_{k+2}$ -equivariant maps

### 4.1 Definitions and preliminaries

Let us define hyperbolicity of non-invertible maps and the notion of Axiom A. See [6] for details. Let  $f$  be a holomorphic map from  $\mathbf{P}^k$  to  $\mathbf{P}^k$  and  $K$  a compact subset such that  $f(K) = K$ . Let  $\widehat{K}$  be the set of histories in  $K$  and  $\widehat{f}$  the induced homeomorphism on  $\widehat{K}$ . We say that  $f$  is hyperbolic on  $K$  if there exists a continuous decomposition  $T_{\widehat{K}} = E^u + E^s$  of the tangent bundle such that  $D\widehat{f}(E_{\widehat{x}}^{u/s}) \subset E_{\widehat{f}(\widehat{x})}^{u/s}$  and if there exists constants  $c > 0$  and  $\lambda > 1$  such that for every  $n \geq 1$ ,

$$|D\widehat{f}^n(v)| \geq c\lambda^n|v| \text{ for all } v \in E^u \text{ and}$$

$$|D\widehat{f}^n(v)| \leq c^{-1}\lambda^{-n}|v| \text{ for all } v \in E^s.$$

Here  $|\cdot|$  denotes the Fubini-Study metric on  $\mathbf{P}^k$ . If a decomposition and inequalities above hold for  $f$  and  $K$ , then it also holds for  $\widehat{f}$  and  $\widehat{K}$ . In particular we say that  $f$  is expanding on  $K$  if  $f$  is hyperbolic on  $K$  with unstable dimension  $k$ . Let  $\Omega$  be the non-wandering set of  $f$ , i.e., the set of points for any neighborhood  $U$  of which there exists an integer  $n$  such that  $f^n(U)$  intersects with  $U$ . By definition,  $\Omega$  is compact and  $f(\Omega) = \Omega$ . We say that  $f$  satisfies Axiom A if  $f$  is hyperbolic on  $\Omega$  and periodic points are dense in  $\Omega$ .

Let us introduce a theorem which deals with repelling part of dynamics. Let  $f$  be a holomorphic map from  $\mathbf{P}^k$  to  $\mathbf{P}^k$ . We define the  $k$ -th Julia set

$J_k$  of  $f$  to be the support of the measure with maximal entropy, in which repelling periodic points are dense. It is a fundamental fact that in dimension 1 the 1st Julia set  $J_1$  coincides with the Julia set  $J$ . Let  $K$  be a compact subset such that  $f(K) = K$ . We say that  $K$  is a repeller if  $f$  is expanding on  $K$ .

**Theorem 6.** ([7]) *Let  $f$  be a holomorphic map on  $\mathbf{P}^k$  of degree at least 2 such that the  $\omega$ -limit set  $E(f)$  is pluripolar. Then any repeller for  $f$  is contained in  $J_k$ . In particular,*

$$J_k = \overline{\{\text{repelling periodic points of } f\}}$$

If  $f$  is critically finite, then  $E(f)$  is pluripolar. We need the theorem above to prove our second result.

## 4.2 Our second result

**Theorem 7.** *For each  $k \geq 1$ , the  $S_{k+2}$ -equivariant map  $g$  satisfies Axiom A.*

*Proof:* We only need to consider the  $S_{k+2}$ -equivariant map  $g$  for a fixed  $k$ , because argument for any  $k$  is similar as the following one. Let us show the statement above for a fixed  $k$  by induction. A restricted map of  $g$  to any  $L^1$  satisfies Axiom A by using the theorem of *critically finite* functions (see [8, Theorem 19.1]). We only need to show that a restricted map of  $g$  to a fixed  $L^2$  satisfies Axiom A. Then a restricted map of  $g$  to any  $L^2$  satisfies Axiom A by symmetry. Argument for a restricted map of  $g$  to any  $L^m$ ,  $3 \leq m \leq k$ , is similar as for a restricted map of  $g$  to the  $L^2$ . Let us denote  $g|_{L^2}$ ,  $\Omega(g|_{L^2})$ , and  $L^2$  by  $g$ ,  $\Omega$ , and  $\mathbf{P}^2$  for simplicity.

We want to show that  $g|_{L^2}$  is hyperbolic on  $\Omega(g|_{L^2})$  by using Kobayashi metrics. If  $g$  is hyperbolic on  $\Omega$ , then  $\Omega$  has a decomposition to  $S_i$ ,

$$\Omega = S_0 \cup S_1 \cup S_2,$$

where  $i=0,1,2$  indicate the unstable dimensions. Since  $C(g)$  attracts all nearby points,  $S_0$  includes all the  $L^0$ 's and  $S_1$  includes all the Julia sets of  $g|_{L^1}$ . We denote by  $J(g|_{L^1})$  the Julia set of  $g|_{L^1}$ . Then  $g$  is contracting in all directions at  $L^0$  and is contracting in the normal direction and expanding in an  $L^1$ -direction on  $J(g|_{L^1})$ . Let us consider a compact, completely invariant subset in  $\mathbf{P}^2 \setminus C$ ,

$$S = \{x \in \mathbf{P}^2 \mid \text{dist}(g^n(x), C) \not\rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

By definition, we have  $J_2 \subset S_2 \subset S$ . If  $g$  is expanding on  $S$ , then it follows that  $S_0 = \cup L^0$ ,  $S_1 = \cup J(g|_{L^1})$ . Moreover  $J_2 = S_2 = S$  holds from Theorem

6 (see Remark 4 below). Since periodic points are dense in  $J(g|_{L^1})$  and  $J_2$ , expansion of  $g$  on  $S$  implies Axiom A of  $g$ .

Let us show that  $g$  is expanding on  $S$ . Because  $f$  is attracting on  $C$  and preserves  $C$ , there exists a neighborhood  $V$  of  $C$  such that  $V$  is relatively compact in  $g^{-1}(V)$  and the complement of  $V$  is connected. We assume one of  $L^1$ 's to be the line at infinity of  $\mathbf{P}^2$ . By letting  $B$  be  $\mathbf{P}^2 \setminus V$  and  $U$  one of connected components of  $g^{-1}(\mathbf{P}^2 \setminus V)$ , we have the following inclusion relations,

$$U \subset g^{-1}(B) \Subset B \subset \mathbf{C}^2 = \mathbf{P}^2 \setminus L^1.$$

Because  $B$  and  $U$  are in a local chart, there exists a constant  $\rho < 1$  such that

$$K_B(x, v) \leq \rho K_U(x, v) \text{ for all } x \in U, v \in T_x \mathbf{C}^2.$$

In addition, since the map  $g$  from  $U$  to  $B$  is an unbranched covering,

$$K_U(x, v) = K_B(g(x), Dg(v)) \text{ for all } x \in U, v \in T_x \mathbf{C}^2.$$

From these two inequalities we have the following inequality

$$K_B(x, v) \leq \rho K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B), v \in T_x \mathbf{C}^2.$$

Since  $g$  preserves  $S$ , which is contained in  $g^{-n}(B)$  for every  $n \geq 1$ ,

$$K_B(x, v) \leq \rho^n K_B(g^n(x), Dg^n(v)) \text{ for all } x \in S, v \in T_x \mathbf{C}^2.$$

Consequently we have the following inequality for  $\lambda = \rho^{-1} > 1$ ,

$$K_B(g^n(x), Dg^n(v)) \geq \lambda^n K_B(x, v) \text{ for all } x \in S, v \in T_x \mathbf{C}^2.$$

Since  $K_B(x, v)$  is upper semicontinuous and  $|v|$  is continuous,  $K_B(x, v)$  and  $|v|$  may be different only by a constant factor. There exists  $c > 0$  such that

$$|Dg^n(x)v| \geq c\lambda^n |v| \text{ for all } x \in S, v \in T_x \mathbf{C}^2.$$

Thus  $g$  is expanding on  $S$  and satisfies Axiom A. □

**Remark 4.** Unlike the case when  $k = 1$ , it does not seem obvious that  $S$  being a repeller implies  $J_k = S$  when  $k \geq 2$ .

**Remark 5.** From [1, Theorem 4.11] and [9], it follows that the Fatou set of the  $S_{k+2}$ -equivariant map  $g$  has full measure in  $\mathbf{P}^k$  for each  $k \geq 1$ .

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## References

- [1] R. BOWEN, "Equilibrium states and the ergodic theory of Anosov diffeomorphisms", Lecture Notes in Mathematics **470**, Springer-Verlag, Berlin-New York, 1975.
- [2] S. CRASS, Solving the sextic by iteration: a study in complex geometry and dynamics, *Experiment. Math.* **8(3)** (1999), 209-240.
- [3] S. CRASS, A family of critically finite maps with symmetry, *Publ. Mat.* **49(1)** (2005), 127-157.
- [4] P. DOYLE AND C. MCMULLEN, Solving the quintic by iteration, *Acta Math.* **163(3-4)** (1989), 151-180.
- [5] J. E. FORNÆSS AND N. SIBONY, Complex dynamics in higher dimension. I. Complex analytic methods in dynamical systems (Rio de Janeiro, 1992), *Astérisque* **222(5)** (1994), 201-231.
- [6] M. JONSSON, Hyperbolic dynamics of endomorphisms, preprint.
- [7] K. MAEGAWA, Holomorphic maps on  $\mathbf{P}^k$  with sparse critical orbits, submitted
- [8] J. MILNOR, "Dynamics in one complex variable", Introductory Lectures, Friedr. Vieweg and Sohn, Braunschweig, 1999.
- [9] M. QIAN AND Z. ZHANG, Ergodic theory for Axiom A endomorphisms, *Ergodic Theory Dynam. Systems* **15(1)** (1995), 161-174
- [10] T. UEDA, Critical orbits of holomorphic maps on projective spaces, *J. Geom. Anal.* **8(2)** (1998), 319-334.
- [11] S. USHIKI, Julia set with polyhedral symmetry, in "Dynamical systems and related topics" (Nagoya, 1990), Adv. Ser. Dynam. Systems **9**, World Sci. Publ., River Edge, NJ, 1991, pp. 515-538.

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